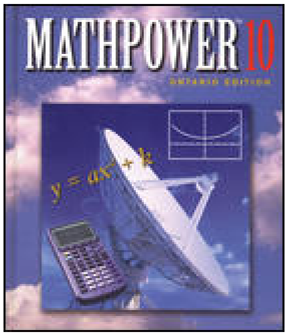
MPM 2D - Grade Ten Academic Mathematics

This guide has been organized in alignment with the 2005 Ontario Mathematics Curriculum. Each of the specific curriculum expectations are cross-referenced to the text book sections.

Contents

Introduction

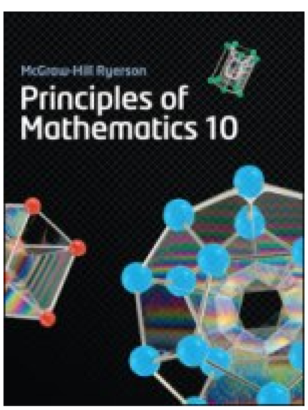
This guide has been arranged in the order of which topics are presented in the Ontario Revised Mathematics Curriculum of 2005. Secondary schools in the Niagara Catholic District School Board at present use the text Math Power 10 by McGraw-Hill Ryerson.



The text has a website with student supports and additional resources that the student may find helpful. It can be accessed through the following web address:

http://www.mcgrawhill.ca/school/booksites/mathpower+10/

This guide was created in this fashion in the event a class uses a different primary resource, such as a textbook from another publisher, or a textbook alternative such as TIPS4RM.



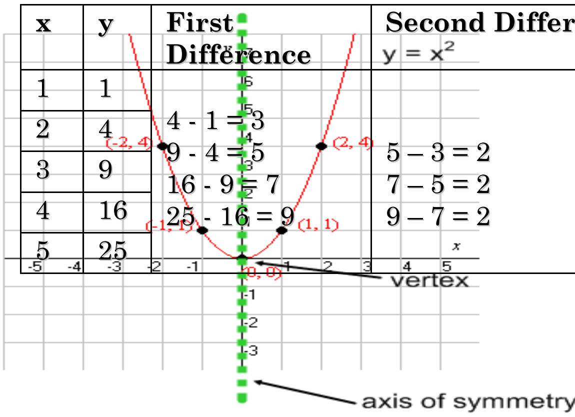
Quadratic Relations

Key Words:	
Algebraic Expression: a mathematical phrase made up of numbers and variables, connected by addition or subtraction signs (eg. $3x + 4$) Also called a **Polynomial**.	**Axis of Symmetry:** the line of symmetry that passes through the vertex of the parabola
Constant Term: a term with only a numerical coefficient and no variable	**Decomposition:** a factoring technique used on trinomials of the form $ax^2 + bx + c$
Degree of a Polynomial: the degree of the highest degree term. Quadratics are degree two polynomials.	**Distributive Property:** rule by which polynomials are multiplied, or **expanded**
Difference of Squares: a special quadratic of the form $a^2 – b^2$	**Domain:** set of first elements in a relation
Factoring: describes an algebraic process of converting a polynomial into a product of two polynomials. Opposite to expanding.	**Function:** a relation in which for every x value there is only one y value
Perfect Square Trinomial: trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$	**Parabola:** the graph of a quadratic relation
Quadratic formula: formula that can be used to find the roots of a quadratic equation	**Range:** set of second elements in a relation
Relation: a set of ordered pairs	**Roots:** the solutions to a quadratic equation
Term: an expression formed by the product of a number and a variable. Polynomials are given names base on the number of terms it has.	**Variable:** a quantity whose value can change, usually represented by a letter, also called a **literal coefficient**
Vertex: the turning point of the parabola	**Vertical Line Test:** to determine if a relation is a function, if any vertical line passes through more than one point on the graph, then the relation is not a function

Properties of Quadratic Relations

- Identify the key features of a parabola

x	y	First Difference	Second Difference	
1	1			**In a quadratic relation, the second differences will all be equal**
2	4	4 - 1 = 3		
3	9	9 - 4 = 5	5 – 3 = 2	
4	16	16 - 9 = 7	7 – 5 = 2	
5	25	25 - 16 = 9	9 – 7 = 2	



- **Finite differences**

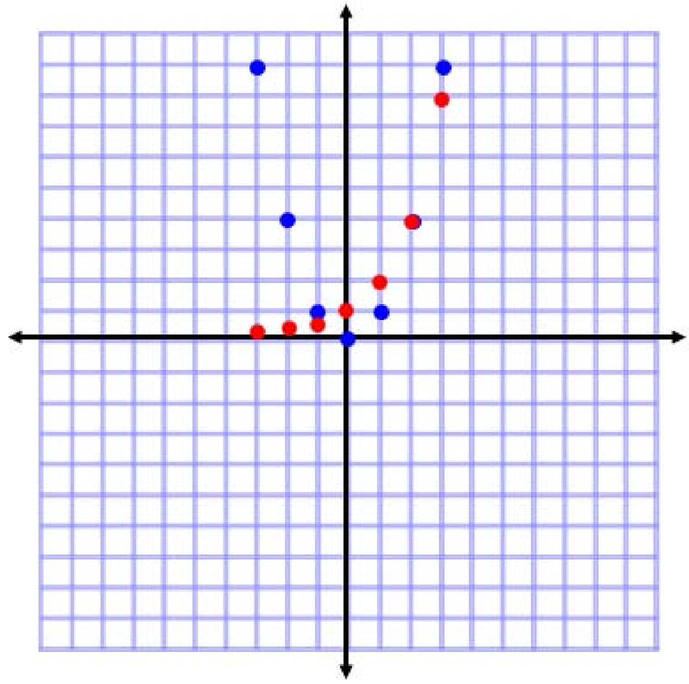
- Comparing $y = x^2$ to $y = 2^x$ and discovering the meaning of zero and negative exponents

$y = x^2$

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

$y = 2^x$

x	y
-3	1/8
-2	1/4
-1	1/2
0	1
1	2
2	4
3	8



- **By observing the patterning in the y value column of $y = 2^x$, we can see the rule for negative and zero exponents:**

$$x^0 = 1 \quad \text{and} \quad x^{-n} = \frac{1}{x^n}$$

Using transformations to relate graphs of quadratic to $y = x^2$

- **$y = a(x-h)^2 + k$**

Students will learn about each of the components in the above standard form of a quadratic and what impact they will have on the graph of $y = x^2$ from a transformation point of view.

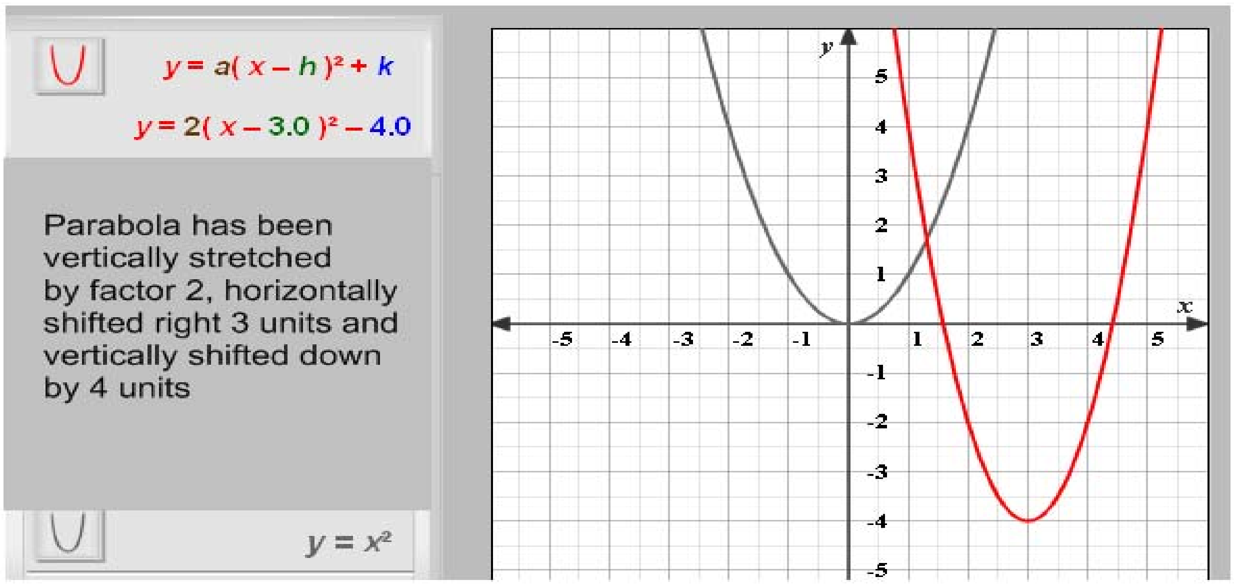
a : the vertical stretch/compression factor and vertical reflection component

if $a > 1$, then parabola is vertically stretched
$0 < a < 1$, then parabola is vertically compressed
$-1 < a < 0$, then parabola is vertically compressed and vertically reflected in the x axis
$a < -1$, then parabola is vertically stretched and vertically reflected in the x axis

h : the horizontal shift factor
if h > 0, then parabola is shifted right h units
h < 0, then parabola is shifted left h units

k : the vertical shift factor
if k > 0, then parabola is shifted up k units
k < 0, then parabola is shifted down k units

Example:



- **The impact the transformations will have on the domain and range, vertex and axis of symmetry are examined.**
  vertex is (h,k)
  axis of symmetry is x = h
  if a > 0, then parabola opens up with a minimum value of k, occurring when x = h
  if a < 0, then parabola opens down with a maximum value of k, occurring when x = h.

Solving Quadratic Equations

- **Polynomials**
  Expand and Simplify second degree polynomial expressions such as:

  $(3x + 4)(2x - 5)$ $\quad$ $(4x - 3)^2$ $\quad$ $(5x - y)(x + 3y)$

Factor Polynomial expressions

Several factoring techniques are covered:

Common Factoring (factoring a G.C.F.) Example: $10x^2 + 15x$
$= 5x(2x + 3)$

5x is the greatest common factor between the two terms

Factoring Trinomials of the Form $x^2 + bx + c$
(Where the number in front of *x* squared is 1)

Example: $x^2 + 9x + 20$
$= (x \underline{+ 4})(x \underline{+ 5})$

Look for a pair of integers so their sum is b & product is c

Factoring Trinomials of the Form $ax^2 + bx + c$
(where *a* does not equal 1)

Example: $2x^2 + 11x + 15$

Look for a pair of integers so their sum is b & product is ac

$= 2x^2 \ \underline{+ 6x} \ \ \underline{+ 5x} + 15$ Decomposition of 11x
$= 2x\,(x + 3) \ +5\,(x + 3)$ GCF factor in each pair
$= \underline{(x + 3)}\,(2x + 5)$ Binomial common factor

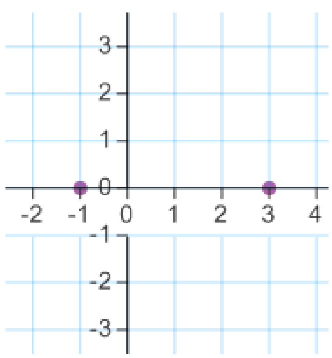
Difference of Squares

Example: $x^2 - 49$
$= (x + 7)(x - 7)$

- **Use factoring to make connections to the zeros/x intercepts of a quadratic in the form $y = a(x-r)(x-s)$**

For Example: Find the x intercepts and sketch the quadratic $y = x^2 - 2x - 3$

$\mathbf{y = x^2 - 2x - 3}$ $= (x - 3)(x + 1)$



So the x intercepts of the graph will be x = 3 and x = -1

Axis of symmetry must lie exactly half way between these points, so vertex must occur at x = 1.

Sub x = 1 into the equation and get y = -4, so vertex has coordinates (1, -4)

Since a = 1, parabola opens up and has not been stretched or compressed

- **Completing the Square**

Converting $y = ax^2 + bx + c$ into $\mathbf{y = a(x-h)^2 + k}$

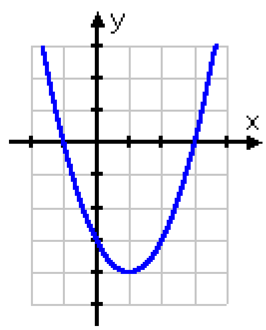
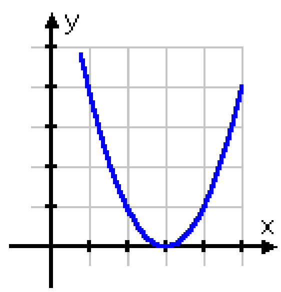
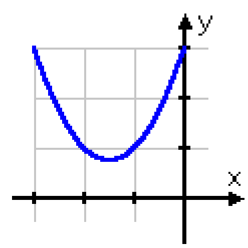
$y = x^2 + 10x + 16$
$y = (x^2 + 10x) + 16$ ignore the constant term, grouping the x terms
$y = (x^2 + 10x \mathbf{+ 25}) \mathbf{- 25} + 16$ take half of the coefficient of the x term, square it and add it and subtract it simultaneously
$y = \mathbf{(x^2 + 10x + 25)} - 25 + 16$ group the first three terms and factor into its perfect square and simplify the constants
$y = \mathbf{(x + 5)^2} - 9$

This skill is used to develop algebraically the quadratic formula. *Although the students are required to reproduce the general case below, it is often shown to help connect the steps to numerical examples.

- **Derive the Quadratic Formula by solving $ax^2 + bx + c = 0$.**

This is the original equation.	$ax^2 + bx + c = 0$
Move the loose number to the other side.	$ax^2 + bx = -c$
Divide through by whatever is multiplied on the squared term. Take half of the x-term, and square it. Add the squared term to both sides.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$ $\frac{b}{2a} \rightarrow \frac{b^2}{4a^2}$ $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$
Simplify on the right-hand side; in this case, simplify by converting to a common denominator.	$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$
Convert the left-hand side to square form (and do a bit more simplifying on the right).	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Square-root both sides, remembering to put the "±" on the right.	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$
Solve for "x =", and simplify as necessary.	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

An understanding is developed between the solutions of a quadratic equation, solved using the quadratic formula and the nature of the graph of the corresponding quadratic relation:

$x^2 - 2x - 3$	$x^2 - 6x + 9$	$x^2 + 3x + 3$
$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)}}{2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = \frac{-2}{2}, \frac{6}{2} = -1, 3$	$x = \frac{6 \pm \sqrt{(-6)^2 - 4(9)}}{2} = \frac{6 \pm \sqrt{36-36}}{2} = \frac{6 \pm \sqrt{0}}{2} = \frac{6 \pm 0}{2} = 3$	$x = \frac{-3 \pm \sqrt{(3)^2 - 4(3)}}{2} = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm \sqrt{-3}}{2} = -\frac{3}{2} \pm \frac{\sqrt{3}i}{2}$
a positive number inside the square root	zero inside the square root	a negative number inside the square root
two real solutions	one (repeated) real solution	two complex solutions
		

Students need only know that this is a non real number

Leading to two non real solutions and no x intercepts

Solving Problems involving Quadratics

Sample Problem:
The path of a basketball shot through the air can be modelled by the equation $h = -0.09d^2 + 0.9d + 2$, where h is height in metres and d is the horizontal distance of the ball from the player in metres.

a) Determine the maximum height of the ball.
b) Determine the horizontal distance of the ball from the player when it is at its maximum height.
c) Determine the height of the ball the moment it is released by the player.

Solutions: Problems such as the one above allows for several approaches, including algebraic, completing the square; graphing using technology, such a graphing calculator or software program; and spreadsheet approach, creating a table of values

Analytic Geometry

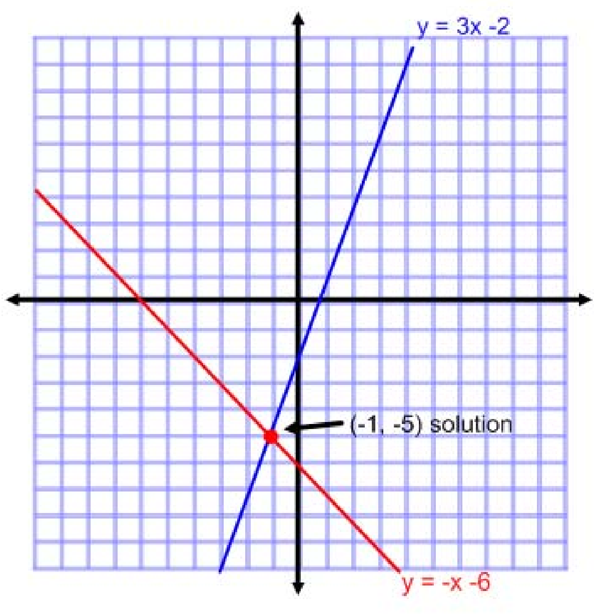
Key Words:

Circumcentre of a Triangle: point at which all three perpendicular bisectors of the three sides of a triangle intersect	**Linear System:** a set of two or more linear equations considered simultaneously
Negative Reciprocals: the slopes of perpendicular lines are negative reciprocals to each other, the product of which is negative one	**Parallel Lines:** lines that run in the same direction and never cross
Perpendicular Bisector: a line that intersects at a right angle and bisects the line segment	**Point of Intersection:** the solution to a linear system, the point at which the two lines cross
Point-Slope Form of a Linear Equation: $y - y_1 = m(x - x_1)$	**Standard Form of a Linear Equation:** $\mathbf{Ax + By + C = 0}$

Using Linear Systems to Solve Problems

- A "system" of equations is a set or collection of equations that are dealt with simultaneously. Linear systems of equations can be solved both graphically and algebraically.
  - Graphically : Find the point of intersection of the two lines.

$$y = 3x - 2 \qquad y = -x - 6$$



  - Algebraically : The methods of substitution and elimination are learned.

In **substitution**, one equation is rearranged and substituted into the other to create an equation in one variable that can be solved for.

Example: $2x - 3y = -2$
$4x + y = 24$

In the equation $4x + \ y = 24$, rearrange to isolate for y and get $y = -4x + 24$

This is then substituted into the first equation, $2x - 3(-4x + 24) = -2$

This equation is now in one variable and can be solved for x. ($x = 5$)

Now this value for x can be substituted into either of the two original equations to find y.

In **elimination**, equations are adjusted so that by adding or subtracting the equations, one of the two variables is eliminated. This will create an equation in one variable that can be solved for.

Example: $2x - 3y = -2$
$4x + \ y = 24$

If the second equation is multiplied by 3, the result would be $12x + 3y = 72$

If this is now added to the first equation,

$$\begin{array}{r} 2x - 3y = -2 \\ \underline{12x + 3y = 72} \\ 14x \quad = 70 \end{array}$$

(the y terms are eliminated)

This equation is now in one variable and can be solved for x. ($x = 5$)

Now this value for x can be substituted into either of the two original equations to find y.

Sample problem: The Robotics Club raised $5000 to build a robot for a future competition. The club invested part of the money in an account that paid 4% annual interest, and the rest in a government bond that paid 3.5% simple interest per year. After one year, the club earned a total of $190 in interest. How much was invested at each rate? Verify your result.

Properties of Lines and Line Segments

- Important Formulas: Slope $\dfrac{y_2 - y_1}{x_2 - x_1}$

Length of a line segment $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Mid point of a line Segment $\left(\dfrac{x_2 + x_1}{2}, \dfrac{y_2 + y_1}{2}\right)$

Equation of a circle with centre (0,0) $x^2 + y^2 = r^2$

Using Analytic Geometry to Verify Geometric Properties

Sample Problems: 1. The sides of a triangle have the equations

$y = -\frac{1}{2}x + 1$, $y = 2x + 4$ and $y = -3x - 9$

Verify that the triangle is an isosceles right triangle.

2. Find the perpendicular distance from the point (5, 6) to the line $-2x + 3y + 4 = 0$

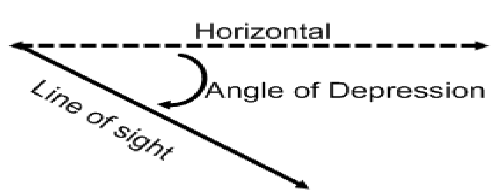
Problem 1 is a straight forward application involving the finding of points of intersection, length of line segments and slope of perpendicular lines.

Problem 2 is a bit more of a challenge. It require you first find the equation of the line that is perpendicular to the given line that passes through the given point. Then the point of intersection of the two lines is found, and then finally the length between the two points is calculated.

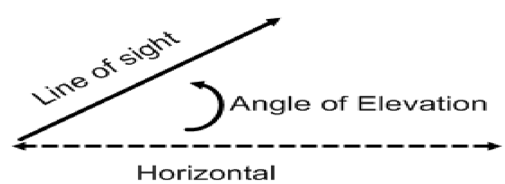
Trigonometry

Key Words:

Angle of Depression: angle formed by the line of site and the horizontal when observing something that is below the horizontal



Angle of Elevation: angle formed by line of site and the horizontal when observing something that is above the horizontal



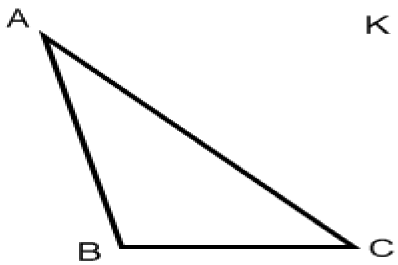
Primary Trig Ratios: ratios sine, cosine and tangent

Pythagorean Theorem: in a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides
Illustrated by the relation $c^2 = a^2 + b^2$

Similar Figures: figures with the same shape, but not necessarily the same size

Similarity

- Similar triangles have all corresponding angles equal and their corresponding side lengths are proportional



∡A = ∡K
∡B = ∡L
∡C = ∡M

$$\frac{AB}{KL} = \frac{BC}{LM} = \frac{AC}{KM}$$

Solving Problems involving Right Triangles and the Pythagorean Theorem

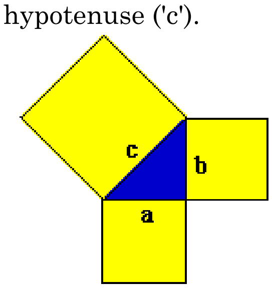
- **The Pythagorean theorem deals with the lengths of the sides of a right triangle.**

  It is often written in the form of the equation:

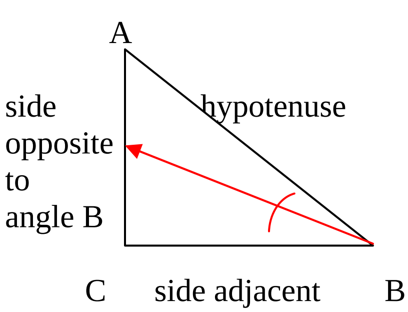
  $a^2 + b^2 = c^2$

  The theorem states that:

  The sum of the squares of the lengths of the legs of a right triangle ('a' and 'b' in the triangle shown below) is equal to the square of the length of the hypotenuse ('c').



- **Primary Trigonometric Ratios**



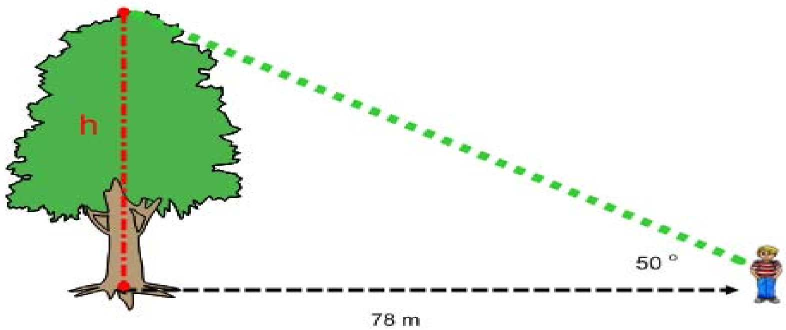
For triangle ΔABC

$\sin B = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\cos B = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\tan B = \dfrac{\text{opposite}}{\text{adjacent}}$

Sample Problem: The angle of elevation of the top of a tree is 50°, for an observer that is standing 78 metres from the base of the tree. Determine the height of the tree.



$\tan 50^\circ = \frac{h}{78}$

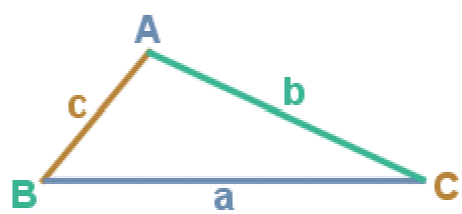
$h = 78 \times \tan 50^\circ$

$h \approx 94$ m

Solving Problems using the Trigonometry of Acute Triangles

- Sine Law and Cosine Law

If we have this triangle:



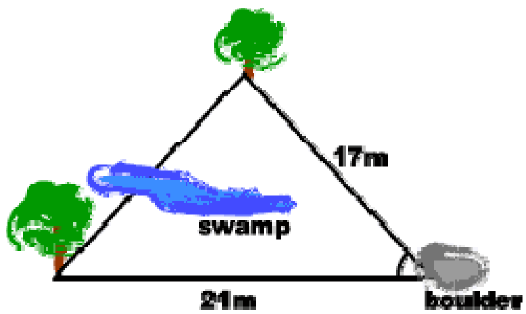
The Sine Law states:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Cosine Law states:

$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

Sample Problem: You want to find the distance between 2 trees (as in the diagram below). Unfortunately, there is a swamp that runs between them which would make it messy to measure the distance directly. However, you know each of their distances from a boulder, as well as the angle made from the boulder to each of the trees (32.98°). Using this information, what is the distance between the trees?



- To solve a triangle is to determine values of all three sides and all three angles

Tips to Solving

1. If the triangle has a right angle, then use it - that is usually much simpler.
2. If the triangle has no right angle, then the type of triangle will determine whether we use The Law of Sines or The Law of Cosines.
3. Usually The Law of Sines is easier to use than The Law of Cosines; so, if you have a choice, use the former.

Solve: